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AN ABSTRACT FORM OF THE NONLINEAR CAUCHY-KOWALEWSKI THEOREM

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Introduction

Consider the initial value problem for functions $u(t, x)$:

$$(1) \qquad \partial_t^m u = f(t, x, u, \partial_x^\alpha \partial_t^j u) , \quad \partial_t^k u|_{t=0} = \phi_k(x); \quad k = 0, \cdots, m-1.$$

Here $x \in \Omega \subset R^n$, $t \in R^1$, and u may be vector valued $u = (u^1, \cdots, u^N)$; f is a nonlinear (N-vector) function depending on t, x, u and all of its derivatives of order $\leq m$ of the form $\partial_x^\alpha \partial_t^j u$, $|\alpha| + j \leq m$, $j < m$. If f is analytic in all its arguments and if the ϕ_k are analytic, then the Cauchy-Kowalewski theorem asserts the existence of a unique analytic solution in a neighborhood of any initial point $(x_0, 0)$.

In the case of linear system of equations several people have observed independently that it is not necessary to assume analyticity in t, i.e., if f is merely continuous in t (with values as an analytic function of the other variables), there exists a unique solution $u(t, x)$ continuously differentiable in t with values in analytic functions of x—in a neighborhood of $(x_0, 0)$. This result has been put into a general, abstract, framework by T. Yamanaka [8] and again by L. V. Ovsjannikov [5] (see J. F. Treves [6] for an exposition and many applications). This result and its proof are direct extensions of the corresponding result and proof for equations with coefficients independent of t of Gelfand, Silov [2]; it is described below in Theorem A.

In [7] (see also [1]) Treves has presented a nonlinear form of the abstract Cauchy-Kowalewski theorem; it is not strong enough, however, to prove existence (and uniqueness) for (1) in the case that f is only continuous in t as an analytic function of the other variables. In this paper we present a nonlinear form of the abstract result which can be applied to this case. After completion of this work we learned that, in fact, this case had been solved by M. Nagumo [4] in 1941. Our result is stated in § 1 and proved in § 2; for completeness the application to (1) is then presented in § 3. Our proof makes use of Newton's iteration method and follows the ideas of J. Moser [3]. In § 4 we also present an implicit function theorem which is essentially just an abstract setting of a

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method in [3]. Though we have given no other applications besides the derivation of Nagumo's result we hope the abstract theorem will find other uses.

Throughout the paper we shall operate within the following framework: X_s is a one parameter family of Banach spaces, where the parameter s varies over the halfopen unit interval $0 \leq s < 1$ (in § 4 we have $0 \leq s \leq 1$). For simplicity it is assumed that all X_s for $s > 0$ are linear subspaces of X_0. It is assumed that

$$(2) \qquad \begin{array}{c} X_s \subset X_{s'} \qquad \textit{for } s' \leq s, \\ \textit{and the natural injection } X_s \to X_{s'} \textit{ has norm } \leq 1. \end{array}$$

$\| \; \|_s$ denotes the norm in X_s. (The space X_0 is not required to be the union of X_s for $s > 0$.)

The variable t will be real or complex, and we shall consider differentiable functions of t in some open neighborhood of the origin with values in one (or more) of the Banach spaces X_s. If t is a complex variable, "differentiable" will mean holomorphic. We propose to study, under appropriate hypotheses, a Cauchy problem of the form

$$(3) \qquad du/dt = F(u(t), t) \; , \qquad |t| < \delta \; ,$$

$$(4) \qquad u(0) = 0 \; .$$

We remark that our results may be easily extended to the case where $u(0) = u_0$ is given—not necessarily zero.

We now describe (as in [6]) the abstract linear Cauchy-Kowalewski theorem of [8], [5] in which

$$F(u, t) = A(t)u + f(t) \; ,$$

where $A(t)$ is continuous in t for $|t| < \eta$ (holomorphic if t is complex)—as a map of X_s to $X_{s'}$ for every s', s in $0 \leq s' < s < 1$, and satisfies

$$(5) \qquad \|A(t)v\|_{s'} \leq C \|v\|_s/(s - s') \qquad \text{for } s' < s \; ,$$

$f(t)$ is a continuous function of t, $|t| < \eta$ (respectively holomorphic) in every X_s for $0 \leq s < 1$. Here C is a fixed constant.

Theorem A. *Let $A(t)$ and $f(t)$ satisfy the above conditions, and set $\delta_0 = \min(\eta, (Ce)^{-1})$. Then, for every s in $0 \leq s < 1$, there is a C^1 (respectively holomorphic) function $u(t)$ of t in $|t| < \delta_0(1 - s)$, with values in X_s, satisfying*

$$(2)' \qquad du/dt = A(t)u(t) + f(t) \; , \qquad u(0) = 0 \; .$$

Furthermore, for any s in $0 \leq s < 1$, there is at most one C^1 solution $u(t)$ in $|t| < \eta$ with values in X_s.

In our treatment of the nonlinear form, Theorem 1.1, we shall make use of a slightly weaker version of Theorem A, Theorem 1.2, which is proved in § 1.

The author wishes to thank J. F. Treves for suggesting this problem.

1. The nonlinear abstract Cauchy-Kowalewski theorem

We consider the problem (3), (4) under the following conditions on F:

(1.1) *For some numbers* $R > 0$, $\eta > 0$, *and every pair of numbers* s, s' *such that* $0 \leq s' < s < 1$, $(u, t) \to F(u, t)$ *is a continuous mapping of*

$$\{u \in X_s;\ \|u\|_s < R\} \times \{t;\ |t| < \eta\} \qquad \text{into } X_{s'} . \tag{1.2}$$

When t is a complex variable, this must be strengthened as follows:

(1.3) *If* $0 \leq s' < s < 1$, *and* $u(t)$ *is a holomorphic function of* t, $|t| < \eta$, *valued in* X_s *such that* $\|u(t)\|_s < R$ *for all* t, $|t| < \eta$, *then* $F(u(t), t)$ *is a holomorphic function of* t, $|t| < \eta$, *valued in* $X_{s'}$.

In addition we assume, and here always $0 \leq s' < s < 1$: *For any positive* $s < 1$ *and every* $u \in X_s$ *with* $\|u\|_s < R$, *and for any* t, $|t| < \eta$, *there is a linear operator* $A_u(t)$ *mapping* X_s *into* X'_s *with*

$$\|A_u(t)v\|_{s'} \leq C\, \|v\|_s/(s - s') \qquad \text{for every } s' < s , \tag{1.4}$$

such that for $\|v\|_s < R$,

$$\|F(v, t) - F(u, t) - A_u(t)(v - u)\|_{s'} \leq C\, \|v - u\|_s^{1+\delta}/(s - s') . \tag{1.5}$$

This is to hold for every $s' < s$, *and with fixed positive constants* $\delta \leq 1$ *and* C, *independent of* t, u, v, s *or* s'.

Finally: $F(0, t)$ *is a continuous function of* t, $|t| < \eta$, (*holomorphic when* t *is complex*) *with values in* X_s *for every* $s < 1$ *and satisfying with a fixed constant* K,

$$|F(0, t)|_s \leq K/(1 - s) , \qquad 0 \leq s < 1 . \tag{1.6}$$

Theorem 1.1. *Under the preceding hypotheses there is a positive number* a *such that there exists a unique function* $u(t)$ *which, for every positive* $s < 1$ *and* $|t| < a(1 - s)$, *is a continuously differentiable function of* t *with values in* X_s; $\|u(t)\|_s < R$, *and* $u(t)$ *satisfies* (1.2), (1.3).

When t is a complex variable "continuously differentiable" means holomorphic.

Before proving Theorem 1.1 we first treat the linear case, a slightly modified form of Theorem A, in which

$$F(u, t) = A(t)u + f(t) \tag{1.7}$$

with $A(t)$ and $f(t)$ satisfying the conditions of Theorem A.

Theorem 1.2. *Let F be as in* (1.7) *and assume* (5) *holds. Let $a \le 1/(8C)$ be a fixed number and suppose that f satisfies for every $s < 1$*

$$\left\| \int_0^t f(\tau)d\tau \right\|_s < k(a(1-s)/|t| - 1)^{-1} \qquad \text{for } |t| < a(s-1) \, . \tag{1.8}$$

Then there is a unique function $u(t)$ which, for every positive $s < 1$ and $|t| < a(1-s)$, is a continuously differentiable function of t with values in X_s, and which satisfies

$$(du/dt)(t) = A(t)u(t) + f(t) \, , \qquad u(0) = 0 \, . \tag{1.9}$$

Furthermore $u(t)$ satisfies

$$\|u(t)\|_s \le 2k(a(1-s)/|t| - 1)^{-1} \qquad \text{for } |t| < a(1-s) \, . \tag{1.10}$$

Remark. If $\|f(t)\|_s \le k/a(1-s)$ for $0 \le s < 1$, then (1.8) holds.

Proof of Theorem 1.2. Our proof is a modification of the usual proofs of Theorem A.

Let B be the space of functions $u(t)$ which, for every nonnegative $s < 1$ and $|t| < a(1-s)$, are continuous functions of t with values in X_s such that

$$M[u] = \sup_{\substack{|t|<a(1-s) \\ 0 \le s < 1}} \|u(t)\|_s (a(1-s)/|t| - 1) < \infty \, . \tag{1.11}$$

B is a Banach space with $M[u]$ as norm. We shall find the solution $u(t)$ as a fixed point in B of the transformation

$$T(v)(t) = \int_0^t (A(\tau)v(\tau) + f(\tau))d\tau;$$

i.e., we show that T maps B into B and is contracting. Hence T has a unique fixed point $u(t)$ in B which is then, clearly, a solution of (1.9).

We first verify the contraction property by showing that if

$$w(t) = \int_0^t A(\tau)v(\tau)d\tau \, ,$$

then

$$M[w] \le \tfrac{1}{2} M[v] \, . \tag{1.12}$$

For $|t| < a(1-s)$ we have, supposing say $t > 0$,

$$\|w(t)\|_s \le \int_0^t \|Av\|_s(\tau)d\tau \le c\int_0^t \frac{\|v\|_{s(\tau)}}{s(\tau)-s}d\tau \,,$$

for some choice of $s(\tau) < 1 - \tau/a$

$$\le cM[v]\int_0^t \frac{d\tau}{(s(\tau)-s)(a(1-s(\tau))/\tau - 1} \,. \tag{1.13}$$

Choose $s(\tau)$, with $\tau < a(1-s(\tau))$ so as to maximize the denominator, in the integral, i.e.,

$$s(\tau) = \tfrac{1}{2}(1 + s - \tau/a) \,.$$

With this choice, $s(\tau) < 1 - \tau/a$, and

$$s(\tau) - s = \tfrac{1}{2}(1 - s - \tau/a) \,,$$

$$a(1 - s(\tau))/\tau - 1 = \tfrac{1}{2}a(1 - s + \tau/a)/\tau - 1 = \tfrac{1}{2}a(1 - s - \tau/a)/\tau \,.$$

Therefore

$$\begin{aligned}\int_0^t \frac{d\tau}{(s(\tau)-s)(a(1-s(\tau))/\tau - 1)} &= \frac{4}{a}\int_0^t \frac{\tau d\tau}{(1-s-\tau/a)^2} \\ &\le 4at\int_0^t (a(1-s)-\tau)^{-2}d\tau = 4at\left[\frac{1}{a(1-s)-t} - \frac{1}{a(1-s)}\right] \\ &= \frac{4t^2}{(1-s)[(1-s)a - t]} = \frac{4t}{(1-s)(a(1-s)/t - 1)} \\ &\le \frac{4a}{(a(1-s)/t - 1)}\,, \qquad \text{since } |t| < a(1-s) \,.\end{aligned} \tag{1.14}$$

Inserting this into (1.13) we find

$$M[w] \le 4aCM[v] \le \tfrac{1}{2}M[v] \,.$$

Next, to see that B is mapped into itself, we note that for $u = Tv$,

$$M[u] \le \tfrac{1}{2}M[v] + M\int_0^t f(\tau)d\tau \le \tfrac{1}{2}M[v] + k < \infty \,.$$

Hence T has a unique fixed point u in B; from the preceding inequality it follows that $M[u] \le 2k$ which is (1.10).

Our fixed point $u(t)$ is a solution of (1.9). To prove uniqueness, suppose $v(t)$ is another solution with $v(t)$ in X_s for $|t| < a(1-s)$; then $w = u - v$ satisfies

$$w(t) = \int_0^t A(\tau)w(\tau)d\tau \ .$$

We cannot use (1.12) to prove $w = 0$ since we do not know that $M[v]$ is finite. To show that $w(t_0) = 0$ as an element in X_s, for fixed t_0 in $|t_0| < a(1 - s)$, let $s < s_0 < 1$ so that $|t_0| < a(s_0 - s)$. Then

$$M_0[w] = \sup_{\substack{|t|<a(s_0-s)\\ s<s_0}} \|w(t)\|_s \left(\frac{a(s_0 - s)}{|t|} - 1 \right) < \infty \ ,$$

and repeating the previous argument we find that

$$M_0[w] \leq \tfrac{1}{2} M_0[w] \ .$$

Hence $M_0[w] = 0$ and so $w(t_0) = 0$—Theorem 1.2 is proved.

2. Proof of Theorem 1.1.

This uses the technique of Moser in § 3 of [3]. We seek a solution of

$$u(t) = \int_0^t F(u(\tau), \tau)d\tau \tag{2.1}$$

with finite norm $M[u]$ defined in (1.10)—but now a will be suitably small. Our solution will be obtained as the limit of a sequence u_k defined recursively by

$$u_0 = 0 \ , \qquad u_{k+1} = u_k + v \ , \tag{2.2}$$

where

$$\|u_k(t)\|_s < R \qquad \text{for } |t| < a_k(1 - s) \ , \tag{2.3}$$

and v is the solution of

$$v(t) = \int_0^t A_{u_k(\tau)}(\tau)v(\tau)d\tau + G_k(t) \tag{2.4}$$

with

$$G_k(t) = \int_0^t F(u_k(\tau), \tau)d\tau - u_k(t) \ . \tag{2.5}$$

Here, for every $s < 1$, and $|t| < a_k(1 - s)$, u_k and $v(t)$ are continuous functions of t with values in X_s for which $M_k[u_k]$ and $M_k[v]$ are finite, where

$$(2.6) \qquad M_k[v] = \sup_{\substack{|t|<a_k(1-s) \\ 0\le s<1}} \|v(t)\|_s \left(\frac{a_k(1-s)}{|t|} - 1\right) < \infty .$$

The numbers a_k will be a decreasing sequence with $a = \lim a_k$. In fact, we shall take

$$(2.7) \qquad a_{k+1} = a_k(1 - (k+1)^{-2}) , \qquad k = 0, 1, \cdots,$$

so that

$$(2.8) \qquad a = a_0 \prod_0^\infty (1 - (k+1)^{-2}) ,$$

and a_0 will be chosen suitably—with $a_0 \le 1$ and $a_0 \le 1/(8C)$.

Let us imagine that u_i are determined for $i \le k$ with $M_i[u_i] < \infty$; set $\lambda_k = M_k[G_k]$. By virtue of (1.4) and Theorem 1.2 there is a function $v(t)$ satisfying (2.4) with

$$(2.9) \qquad M_k[v] \le 2\lambda_k .$$

Hence

$$\|v(t)\|_s \le \frac{2\lambda_k}{a_k/a_{k+1} - 1} \qquad \text{for } |t| < a_{k+1}(1-s) ,$$

and it follows that for $|t| < a_{k+1}(1-s)$

$$\|u_{k+1}(t)\|_s \le 2\frac{\lambda_k}{a_k/a_{k+1} - 1} + \|u_k(t)\|_s ,$$

and so, by recursion,

$$(2.10) \qquad \|u_{k+1}(t)\|_s \le 2 \sum_0^k \lambda_j(a_j/a_{j+1} - 1)^{-1} .$$

We will require that

$$(2.11) \qquad 2 \sum_0^k \lambda_j(a_j/a_{j+1} - 1)^{-1} < R/2 .$$

Then for $|t| < a_{k+1}(1-s)$ we have $\|u_{k+1}(t)\|_s < R/2$, and so $F(u_{k+1}(t), t)$ is defined.

Our aim is to have $\lambda_j \to 0$ as $j \to \infty$. To estimate λ_{k+1}, we have from (2.4) and (2.5),

$$G_{k+1}(t) = \int_0^t F(u_{k+1}(\tau), \tau)d\tau - u_{k+1}(t)$$

$$= \int_0^t [F(u_{k+1}(\tau), \tau) - F(u_k(\tau), \tau) - A_{u_k(\tau)}(\tau)v(\tau)]d\tau \ .$$

Thus for $|t| < a_{k+1}(1 - s)$, we see from (1.5) that

$$\|G_{k+1}(t)\|_s \le C \int_0^t \frac{\|v\|_{s(\tau)}^{1+\delta}}{s(\tau) - s} d\tau$$

for some choice of $s(\tau) < 1 - |\tau|/a_k$. Observing that $s < 1 - |t|/a_{k+1} \le 1 - |t|/a_k < 1 - |\tau|/a_k$ we may set

$$s(\tau) = \tfrac{1}{2}(1 - |\tau|/a_k + s) \ .$$

Then $a_k(1 - s(\tau))/\tau - 1 = \frac{1}{2}(a_k(1 - s) - \tau)/\tau$, and we find with the aid of (2.9), (assuming, say, $t > 0$)

$$\|G_{k+1}(t)\|_s \le C(2\lambda_k)^{1+\delta} \int_0^t \frac{d\tau}{\frac{1}{2}(1 - \tau/a_k - s)(a_k(1 - s(\tau))/\tau - 1)^{1+\delta}}$$

$$= 2^{3+2\delta} C\lambda_k^{1+\delta} a_k \int_0^t \tau^{1+\delta}(a_k(1 - s) - \tau)^{-(2+\delta)} d\tau$$

$$\le 2^5 C\lambda_k^{1+\delta} t^{1+\delta} \int_0^t (a_k(1 - s) - \tau)^{-(2+\delta)} d\tau \ .$$

Now

$$\int_0^t (a_k(1 - s) - \tau)^{-(2+\delta)} d\tau = \frac{1}{1 + \delta}[a_k(1 - s) - t)^{-(1+\delta)} - (a_k(1 - s))^{-(1+\delta)}]$$

$$\le t a_k^{-1}(1 - s)^{-1}(a_k(1 - s) - t)^{-(1+\delta)} \ .$$

Inserting this into the preceding estimate we obtain the inequality

$$\|G_{k+1}(t)\|_s \le 2^5 C\lambda_k^{1+\delta} \frac{t}{a_k(1 - s)} \frac{1}{(a_k(1 - s)/t - 1)^{1+\delta}} \ .$$

Consequently

$$\lambda_{k+1} = M_{k+1}[G_{k+1}] \le 2^5 C\lambda_k^{1+\delta} \sup_{\substack{|t| < a_{k+1}(1-s) \\ s<1}} \left\{ \frac{t}{a_k(1 - s)} \frac{(a_{k+1}(1 - s)/t - 1)}{(a_k(1 - s)/t - 1)^{1+\delta}} \right\}$$

$$= 2^5 C\lambda_k^{1+\delta} \sup_{t < a_{k+1}/a_k} \frac{(a_{k+1}/a_k - t)t^{1+\delta}}{(1 - t)^{1+\delta}} \le 2^5 C\lambda_k^{1+\delta} \sup_{t < a_{k+1}/a_k} \frac{(a_{k+1}/a_k - t)}{(1 - t)^{1+\delta}} \ .$$

One easily finds that the supremum in the preceding is $\dfrac{1}{(1 - a_{k+1}/a_k)^\delta}$, and hence

$$\lambda_{k+1} \le \frac{2^5 C \lambda_k^{1+\delta}}{(1 - a_{k+1}/a_k)^\delta} \, . \tag{2.12}$$

We are now ready to choose a_0. Using (1.6) we observe first that

$$\begin{aligned}\lambda_0 &= M_0\left[\int_0^t F(0, \tau) d\tau\right] \\ &\le K \sup_{|t| < a_0(1-s)} \left[\frac{|t|}{1 - s}\left(\frac{a_0(1 - s)}{|t|} - 1\right)\right] \le a_0 K \, .\end{aligned}$$

We shall require that

$$\lambda_j \le a_0 K (j + 1)^{-4} \, . \tag{2.13}$$

Assuming that this is true for λ_k we find, from (2.12) and (2.7),

$$\begin{aligned}\lambda_{k+1} &\le 2^5 C (a_0 K)^{1+\delta} (k + 1)^{-(4+2\delta)} \\ &\le \frac{a_0 K}{(k + 2)^4}\left[2^5 C (a_0 K)^\delta \frac{(k + 2)^4}{(k + 1)^{4+2\delta}}\right] \le a_0 K (k + 2)^{-4} \, ,\end{aligned}$$

provided $a_0 \le a'$ independent of k.

We have now to verify (2.11). From (2.7) and (2.13)

$$\begin{aligned}2 \sum_0^k \lambda_j (a_j/a_{j+1} - 1)^{-1} &\le 2 \sum_0^k \lambda_j (1 - a_{j+1}/a_j)^{-1} \\ &= 2 \sum_0^k \lambda_j (j + 1)^2 \le 2 a_0 K \sum_0^k (j + 1)^{-2} \\ &\le 2 a_0 K \sum_0^\infty (j + 1)^{-2} < R/2 \quad \text{provided } a_0 \le a'' \, .\end{aligned} \tag{2.14}$$

If we now choose $a_0 \le a'$, $a_0 \le a''$ we find that the functions u_k are defind for all k, with

$$\| u_k(t) \|_s < R/2 \qquad \text{for } |t| < a_k(1 - s) \, . \tag{2.15}$$

Furthermore we have from (2.9)

$$\| u_{k+1}(t) - u_k(t) \|_s \le 2 \lambda_k (a_k(1 - s)/|t| - 1)^{-1} \qquad \text{for } |t| < a_k(1 - s) \, .$$

Hence if $|t| < a(1 - s) < a_k(1 - s)$ with $a = \lim a_k$, then we find

$$\|u_{k+1}(t) - u_k(t)\|_s \leq 2\lambda_k(a(1-s)/|t| - 1)^{-1} ,$$

or

$$M[u_{k+1} - u_k] \leq 2\lambda_k .$$

Since $\sum \lambda_k < \infty$, it follows that u_k converges to some $u(t)$ in B. From (2.15) it follows that $\|u(t)\|_s \leq R/2$ for $|t| < a(1-s)$. We claim, finally, that $u(t)$ is a solution of (2.1).

We have namely for $|t| < a(1-s') < a(1-s)$

$$\begin{aligned}
&\left\| \int_0^t F(u(\tau), \tau) d\tau - u(t) \right\|_{s'} \\
&\qquad \leq \int_0^t \|F(u(\tau), \tau) - F(u_k(\tau), \tau)\|_{s'} \, d\tau + \|u(t) - u_k(t)\|_{s'} \\
&\qquad\quad + \lambda_k(a(1-s)'/t - 1)^{-1} \\
&\qquad \leq \frac{C}{s-s'} \left\{ \int_0^t [\|u(\tau) - u_k(\tau)\|_s + \|u(\tau) - u_k(\tau)\|_s^{1+\delta}] d\tau \right\} \\
&\qquad\quad + \|u(t) - u_k(t)\|_{s'} + \lambda_k(a(1-s')/|t| - 1)^{-1}
\end{aligned}$$

by (1.5). All the terms on the right go to zero as $k \to \infty$, and it follows that $u(t)$ is a solution of (2.1). Clearly $u(t)$ is also a solution of (3), (4).

To complete the proof of Theorem 1.1 we have to prove uniqueness of the solution. Suppose $v(t)$ is also a solution. Then $w(t) = v(t) - u(t)$ satisfies

$$w(t) = \int_0^t [F(u(\tau), \tau) - F(v(\tau), \tau)] d\tau .$$

For any fixed $s_0 < 1$, the functions u and v have finite M_0 norm where

$$M_0[u] = \sup_{\substack{|t| < a(s_0 - s) \\ 0 \leq s < s_0}} \|u(t)\|_s \, (a(s_0 - s)/|t| - 1) .$$

Hence for $|t| < a(s_0 - s)$ we find from (1.5)

$$\|w(t)\|_s \leq C(1 + R^\delta) \left| \int_0^t \frac{\|w(\tau)\|_{s(\tau)} \, d\tau}{s(\tau) - s} \right|$$

for some choice of $s(\tau) \leq s_0 - |\tau|/a$. Arguing as in the proof of Theorem 1.2 we obtain the inequality

$$\|w(t)\|_s \leq 4aC(1 + R^\delta)\frac{M_0[w]}{a(s_0 - s)/|t| - 1}$$

so that $M_0[w] \leq 4aC(1 + R^\delta)M_0[w]$.

Hence we conclude that $M_0[w] = 0$ provided $4aC(1 + R^\delta) < 1$ which we can always assume—by decreasing a if necessary. Thus $\|w(t)\|_s = 0$ for $|t| < a(s_0 - s)$. Since this is true for every s_0 we conclude that $w \equiv 0$, and Theorem 1.1 is proved.

Remark. It is clear from the proof that Theorem 1.1 holds if the condition (1.6) is replaced by the weaker condition:

For some real positive number a', all nonnegative $s < 1$, and $|t| < a'(1 - s)$, $F(0, t)$ is a continuous function of t with values in X_s, satisfying for some constant k

$$\left\| \int_0^t F(0, \tau) d\tau \right\|_s \leq k(a'(1 - s)/|t| - 1)^{-1} . \tag{1.6$'$}$$

3. The nonlinear initial value problem

In this section we treat (1) assuming f to be continuous in t with values in the space of holomorphic (N-vector) functions of the other variables near the origin. In case f is also analytic in t the same proof for a complex neighborhood $|t| < \eta$ yields the classical Cauchy-Kowalewski theorem, and we shall not say any more about that case. By subtraction of a suitable function we may assume that the initial data, the ϕ_j, all vanish.

The first step in the proof of local existence and uniqueness is the standard one of reduction of the problem to an equivalent one for a system of first order (see [6]). If we introduce

$$u_i = \partial u/\partial x_i \ , \quad i = 1, \cdots, n \ , \quad u_{n+1} = \partial u/\partial t \ ,$$

we see that the problem (1) of existence and uniqueness for

$$\partial_t^m u = f(t, x, u, \cdots) \ , \quad \partial_t^k u|_{t=0} = 0 \ , \quad k = 0, \cdots, m - 1$$

is equivalent to the problem for the system for $(u, u_1, \cdots, u_{n+1})$:

$$\begin{aligned} \partial_t^{m-1} u &= \partial_t^{m-2} u_{n+1} \ , \\ \partial_t^{m-1} u_i &= \partial_{x_i} \partial_t^{m-2} u_{n+1} \ , \qquad i \leq n \ , \\ \partial_t^{m-1} u_{n+1} &= f(t, x, u, \cdots) \ , \end{aligned}$$

with $\partial_t^k u = \partial_t^k u_i = 0$ at $t = 0$ for $k < m - 1$, $i - 1, \cdots, n + 1$. Here the derivatives of u in the arguments of f are replaced by derivatives of order at most $m - 1$ of $u, u_1, \cdots, u_{n+1}$. This new problem is of order $M - 1$. Repeating this process we finally obtain a system of first order.

So let us consider (0.1) with $m = 1$:

$$\partial_t u = f(t, x, u, u_{x_1}, \cdots, u_{x_n}) , \qquad u(x, 0) = 0 . \tag{3.1}$$

Here $f(t, x, u, p)$ is continuous in t with values in holomorphic functions of the other arguments for $|x_j| \leq R$, $|u| \leq R$, $|p| \leq R$; $p = (p_1, \cdots, p_n)$ and each p_i is an N-vector.

The next (standard) step is to reduce this to an equivalent first order system which is linear in the derivatives of the unknowns. Introduce

$$p_i = \partial_{x_i} u , \qquad i = 1, \cdots, n ,$$

then existence and uniqueness for (3.1) is equivalent to the same problem for the system for $(u, p_1, \cdots, p_n)$—in obvious notation—

$$\partial_t u = f(t, x, u, p) ,$$
$$\partial_t p_i = f_{x_i}(t, x, u, p) + f_u(t, x, u, p) u_{x_i} + f_p(t, x, u, p) p_{x_i} , \qquad i = 1, \cdots, n ,$$

with $u(x, 0) = p_i(x, 0) = 0$.

Thus it suffices to treat a quasilinear system of the form

$$\partial_t u = \sum a^j(t, x, u) u_{x_j} + b(t, x, u) , \qquad u(x, 0) = 0 \tag{3.2}$$

for an N-vector u; each a^j is an $N \times N$ matrix, and b is an N-vector. The components of a^j and b are continuous in t, for $|t| < \eta$ with values which are holomorphic functions in a neighborhood of $\prod_j \{|x_j| \leq R\} \times \prod_i \{|u^i| \leq R\}$. Here x_j and the components u^i are complex valued. We suppose that a^j and b and their first and second derivatives with respect to the x_k and u^i are bounded by some constant c.

For $0 \leq s < 1$ let X_s denote the space of vector functions $u(x)$ which are holomorphic and bounded in $D_s = \prod_j \{|x^j| < sR\}$, and set

$$\|u\|_s = \sup_{D_s} |u(x)| . \tag{3.3}$$

By the usual estimate for derivatives of holomorphic functions we have

$$\|\partial_{x_j} u\|_{s'} \leq R^{-1} \|u\|_s / (s - s') , \qquad \text{for } 0 \leq s' < s . \tag{3.4}$$

If we denote the operator $a^j(t, x, u) u_{x_j} + b(t, x, u)$ (using summation convention) by $F(u(t), t)$, where $u(t) = u(t, x)$, we see that F satisfies condition (1.1). Also $F(0, t) = b(t, x, 0)$ is bounded and so certainly satisfies (1.6). Next we see with the aid of the mean value theorem that if $|u(x)| < R$ in the region D_s, then, in $D_{s'}$, $s' < s$

$$\begin{aligned}|F(v,t) - F(u,t) - a^j(t,x,u)(v_{x_j} - u_{x_j}) \\ - \sum_i (a^j_{u^i}(t,x,u)u_{x_j} + b_{u^i}(t,x,u))(v^i - u^i)\end{aligned}$$

$$\begin{aligned}&\le C_1 |v-u|^2(|v_x|+1) && \text{for some fixed constant } C_1\\ &\le C_1 \|v-u\|^2_{s'}((s-s')^{-1}+1) && \text{by (3.4)}\\ &\le 2C_1 \|v-u\|^2_{s'}/(s-s') && \\ &\le 2C_1 \|v-u\|^2_{s}/(s-s') && \text{a fortiori .}\end{aligned}$$

Thus we see that (1.5) is satisfied with $C = 2C_1$, $\delta = 1$ and

$$A_u(t)w = a^j(t,x,u(x))w_{x_j} + (a^j_{u^i}(t,x,u)u_{x_j} + b_{u^i}(t,x,u))w^i \, .$$

Furthermore, for $\|u\|_s < R$, we have with a suitable constant C_2

$$\begin{aligned}\|A_u(t)w\|_{s'} &\le C_2 \|w_x\|_{s'} + C_2 R(\|u_x\|_{s'} + 1)\|w\|_{s'}\\ &\le C_2 R^{-1}\|w\|_s/(s-s') + (C_2/(s-s') + C_2 R)\|w\|_{s'}\\ &\le C\|w\|_s/(s-s')\end{aligned}$$

for a suitable constant C.

Thus all the conditions of Theorem 1.1 hold with a suitable constant C, and the local existence and uniqueness of solutions of (3.2) which are analytic in x follows from Theorem 1.1.

4. An implicit function theorem

We present a form of the implicit function theorem following § 3 of [3] set within the abstract framework of our Banach spaces X_s.

Consider two one-parameter families of Banach spaces X_s, Y_s in the closed unit interval: for $0 \le s' < s \le 1$,

$$X_0 \supset X_{s'} \supset X_s \supset X_1 \, , \qquad Y_0 \supset Y_{s'} \supset Y_s \supset Y_1 \, ,$$

and with norms $\| \ \|_s$ in X_s and $| \ |_s$ in Y_s satisfying

$$\|x\|_{s'} \le \|x\|_s \, , \qquad |y|_{s'} \le |y|_s$$

for $x \in X_s$, $y \in Y_s$ and $0 \le s' < s \le 1$.

With R a fixed positive number let $F(u)$ be a mapping into Y_0 which is defined for every u belonging to some X_s satisfying

$$\|u\|_s < R \, , \tag{4.1}$$

and is a continuous map of this ball in X_s into $Y_{s'}$ for every $s' < s$. Our aim is to solve the equation

$$F(u) = 0 \tag{4.2}$$

for u in some X_s—assuming $|F(0)|_1$ is sufficiently small. We make the following hypotheses in which $p, q, C > 0$, $0 < \delta \leq 1$, are fixed:

(i) *For every s in $0 \leq s \leq 1$ and $u, v \in X_s$ with $\|u\|_s, \|v\|_s < R$ there is a linear operator A_u mapping X_σ into $X_{\sigma'}$ for every $\sigma' < \sigma \leq s$ satisfying*

$$|F(v) - F(u) - A_u(v - u)|_{\sigma'} \leq C \|v - u\|_\sigma^{1+\delta}(\sigma - \sigma')^{-p} . \tag{4.3}$$

(ii) *For s in $0 \leq s \leq 1$, any $u \in X_s$ with $\|u\|_s < R$, and any f in Y_σ, $\sigma < s$, there is a solution w, belonging to $X_{\sigma'}$ for every $\sigma' < \sigma$, of the equation*

$$A_u w = f , \tag{4.4}$$

and satisfying

$$\|w\|_{\sigma'} \leq C |f|_\sigma (\sigma - \sigma')^{-q} . \tag{4.5}$$

Theorem 4.1. *Under the above conditions* (i), (ii) *for any nonnegative $s < 1$ there is a number $\varepsilon_0(s)$ such that if $|F(0)|_1 \leq \varepsilon_0(s)$ there is a solution u in X_s of $F(u) = 0$.*

Proof. Set $\rho = (1 - s) \sum_1^\infty k^{-2}$. Associated with the decreasing sequence s_k $(k = 0, 1, \cdots)$ defined by

$$s_0 = 1 , \quad s_{k-1} - s_k = \rho k^{-2} , \quad k > 0,\ s_0 = 1 , \tag{4.6}$$

we define by recursion a sequence $u_k \in X_{s_k}$, $k = 0, 1, \cdots$, starting with $u_0 = 0$ and

$$u_{k+1} = u_k + v , \tag{4.7}$$

where v is a solution in $X_{s'}$ for all $s' < s_k$ of

$$A_{u_k} v = -F(u_k) \tag{4.8}$$

furnished by condition (ii).

In order to ensure that the u_k are well defined, there are several things to be verified. Suppose that $u_0, \cdots, u_k$ have been so defined, satisfying

$$\|u_i\|_{s_i} \leq R/2 \qquad i \leq k . \tag{4.9}$$

Set

$$\tau_i = s_i - \tfrac{1}{2}\rho(i + 1)^{-2} \qquad i = 0, 1, \cdots \tag{4.10}$$

so that

$$\tau_{i-1} - \tau_i = \tfrac{1}{2}\rho(i^{-2} + (i+1)^{-2}) \geq \bar{\rho} i^{-2}\,, \qquad i = 1, 2, \cdots, \text{ for a suitable constant } \bar{\rho}, \tag{4.11}$$

and set

$$\lambda_i = |F(u_i)|_{\tau_i}\,, \qquad i = 0, \cdots, k+1\,. \tag{4.12}$$

For $\sigma < \tau_k$ we have

$$\|v\|_\sigma \leq C\lambda_k(\tau_k - \sigma)^{-q}\,. \tag{4.13}$$

From (4.3) and (4.7), (4.8) we may infer that for $\tau_{k+1} < \sigma < \tau_k$,

$$\begin{aligned} \lambda_{k+1} = |F(u_{k+1})|_{\tau_{k+1}} &\leq C\,\|v\|_\sigma^{1+\delta}(\sigma - \tau_{k+1})^{-p} \\ &\leq C^{2+\delta}\lambda_k^{1+\delta}(\sigma - \tau_{k+1})^{-p}(\tau_k - \sigma)^{-(1+\delta)q} \qquad \text{by (4.13).} \end{aligned} \tag{4.14}$$

If we now set

$$\sigma = \tau_{k+1} + \frac{p}{p + q(1+\delta)}(\tau_k - \tau_{k+1})\,,$$

(this choice maximizes the denominator in (4.14)) we obtain from (4.14)

$$\lambda_{k+1} \leq C_1\lambda_k^{1+\delta}(\tau_k - \tau_{k+1})^{-p-q(1+\delta)}\,,$$

where C_1 is a fixed constant independent of k. Consequently, from (4.11) we find, for some constant C_2 independent of k,

$$\lambda_{k+1} \leq C_2(k+1)^{2(p+q+q\delta)}\lambda_k^{1+\delta}\,. \tag{4.15}$$

Suppose now that we have obtained u_i, $i = 0, \cdots, k$, satisfying (4.9), and, in addition, for some constant ε,

$$\lambda_i \leq \varepsilon(i+1)^{-r}\,, \quad r = 1 + 2(p + q + q\delta)/\delta\,, \quad i = 0, \cdots, k\,. \tag{4.16}$$

Then we find from (4.15)

$$\lambda_{k+1} \leq C_2\varepsilon^{1+\delta}\frac{1}{(k+1)^{1+\delta+2(p+q+q\delta)/\delta}} \leq C_3\varepsilon^{1+\delta}(k+2)^{-r}$$

with a constant C_3 independent of k and ε. Consequently if (4.16) holds with $\varepsilon = |F(0)|_1$ so small that $C_3\varepsilon^\delta < 1$, then we see by recursion that

$$\lambda_{k+1} \leq \varepsilon(k+2)^{-r}\,.$$

We still have to verify that (4.9) holds for $i = k+1$. To this end observe that we have from (4.13) and (4.16)

$$\begin{aligned} \|u_{k+1} - u_k\|_{s_{k+1}} &\le C\lambda_k(\tau_k - s_{k+1})^{-q} = C[2(k+1)^2/\rho]^q\lambda_k \\ &\le \frac{C_4\varepsilon}{(k+1)^{1+2(p+q)/\delta}}, \end{aligned} \tag{4.17}$$

where C_4 is a constant independent of k and ε. Thus, since $u_0 = 0$,

$$\|u_{k+1}\|_{s_{k+1}} \le \sum_{i=1}^{k+1} \|u_i - u_{i-1}\|_{s_i} \le C_4\varepsilon \sum_1^\infty (k+1)^{-1-2(p+q)/\delta} \le R/2$$

for ε sufficiently small.

We have verified that for $\varepsilon = |F(0)|_1$ sufficiently small, the $u_k \in X_{s_k}$ are well defined by recursion and satisfy (4.9) and (4.17). The sequence s_k is decreasing with s as limit. From (4.17) it follows that the sequence u_k converges in X_s to an element u satisfying $\|u\|_s \le R/2$. Since F is a continuous map of the ball $\|x\|_s < R$ in X_s into $Y_{s'}$ for every $s' < s$ and since $F(u_k) \to 0$ in X_s, it follows that $F(u) = 0$.

Added in proof. Recently Ovsjannikov has published a different abstract from of the nonlinear Cauchy problem: L.V. Ovsjannikov, *A nonlinear Cauchy problem in a scale of Banach spaces*, Dokl. Akad. Nauk SSSR **200** (1971) 789–792; Soviet Math. Dokl. **12** (1971) 1497–1502.

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